

Linear Electron Response to An Ion (Analytical Approach)

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Outline

- Introduction
- Assumptions and Equation of motion
- Analytical Solution For $f_0^e(\vec{v})$ =Lorenzian Distribution
 - Space domain solution.
 - Wavelength domain solution (It shows how electron response depends on wavelength).
- Numerical Solution For $f_0^e(\vec{v})$ =Gaussian Distribution
 - Checking codes with analytical formula for Lorenzian distribution (Debug).
 - Comparing results for Gaussian and Lorenzian distribution.

Introduction

- The response of the electron beam to moving ions is important for CEC. The linear response will be amplified to cool the ions and the non-linear response might set limitation to the cooling rate.
- Under certain assumptions, analytical formula can be found to describe the linear response. Numerical solution is relatively straightforward for more general case.

Assumptions

- The electron beam is treated as **an infinite single-species plasma** with 3D temperatures.
- The velocity distribution of the electron beam is **Lorenzian**. (Numerical solution can be found for other velocity distributions.)
- The response has to be weak compared with the background in order to stay in **linear region**.

Equation Of Motion

- Linearized Vlasov Equation

$$\frac{\partial}{\partial t} f_1(\vec{x}, \vec{v}, t) + \vec{v} \cdot \frac{\partial}{\partial \vec{x}} f_1(\vec{x}, \vec{v}, t) - \frac{e \vec{E}}{m_e} \frac{\partial}{\partial \vec{v}} f_0(\vec{v}) = 0$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho(\vec{x}, t)}{\epsilon_0}$$

$$\rho(\vec{x}, t) = eZ\delta(\vec{x} - \vec{v}_0 t) - e \int f_1(\vec{x}, \vec{v}, t) d\vec{v}$$

Defined as $\tilde{n}_1(\vec{x}, t)$

Equation Of Motion Contin.

- Fourier transform the equation of motion to k space.

$$\frac{\partial}{\partial t} f_1(\vec{k}, \vec{v}, t) + i \vec{k} \cdot \vec{v} f_1(\vec{x}, \vec{v}, t) + i \frac{e}{m_e} \Phi(\vec{k}, t) \vec{k} \cdot \frac{\partial}{\partial \vec{v}} f_0(\vec{v}) = 0$$

$$\Phi(\vec{k}, t) = \frac{e}{\epsilon_0 k^2} [Z_i - \tilde{n}_1(\vec{k}, t)]$$

- Integrating the equation over \vec{v} , one gets

$$\tilde{n}_1(\vec{k}, t) = \omega_p^2 \int_0^t [\tilde{n}_1(\vec{k}, t_1) - Z_i] (t_1 - t) g(\vec{k}(t_1 - t)) dt_1$$

$$g(\vec{u}) \equiv \int f_0(\vec{v}) e^{i \vec{u} \cdot (\vec{v} + \vec{v}_0)} d^3 v$$

Ion velocity

Solution For Lorenzian Distribution

- Equilibrium electron velocity distribution

$$f_0(\vec{v}) = \frac{1}{\pi^2 \sigma_x \sigma_y \sigma_z} \frac{1}{\left(1 + \frac{v_x^2}{\sigma_x^2} + \frac{v_y^2}{\sigma_y^2} + \frac{v_z^2}{\sigma_z^2}\right)^2}$$

,which gives

$$g(\vec{u}) = \exp(i\vec{u} \cdot \vec{v}_0) \exp\left(-\sqrt{(u_x \sigma_x)^2 + (u_y \sigma_y)^2 + (u_z \sigma_z)^2}\right)$$

- For Lorenzian distribution, the integral equation reduced to an 2nd order ODE

$$\ddot{H}(\vec{k}, t) = -\omega_p^2 H(\vec{k}, t) + Z_i \omega_p^2 e^{-\lambda(\vec{k})t}$$

$$H(\vec{k}, t) \equiv \tilde{n}_1(\vec{k}, t) \exp[-\lambda(\vec{k}) \cdot t]$$

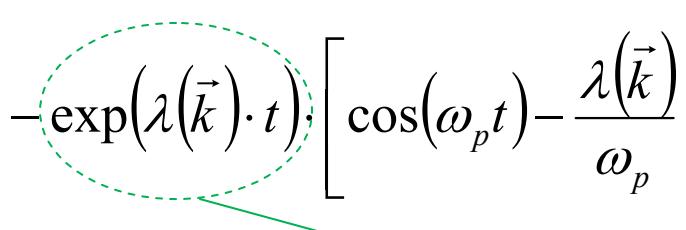
Solution in K space

where

$$\lambda(\vec{k}) \equiv i\vec{k} \cdot \vec{v}_0 - \sqrt{(k_x \sigma_x)^2 + (k_y \sigma_y)^2 + (k_z \sigma_z)^2}$$

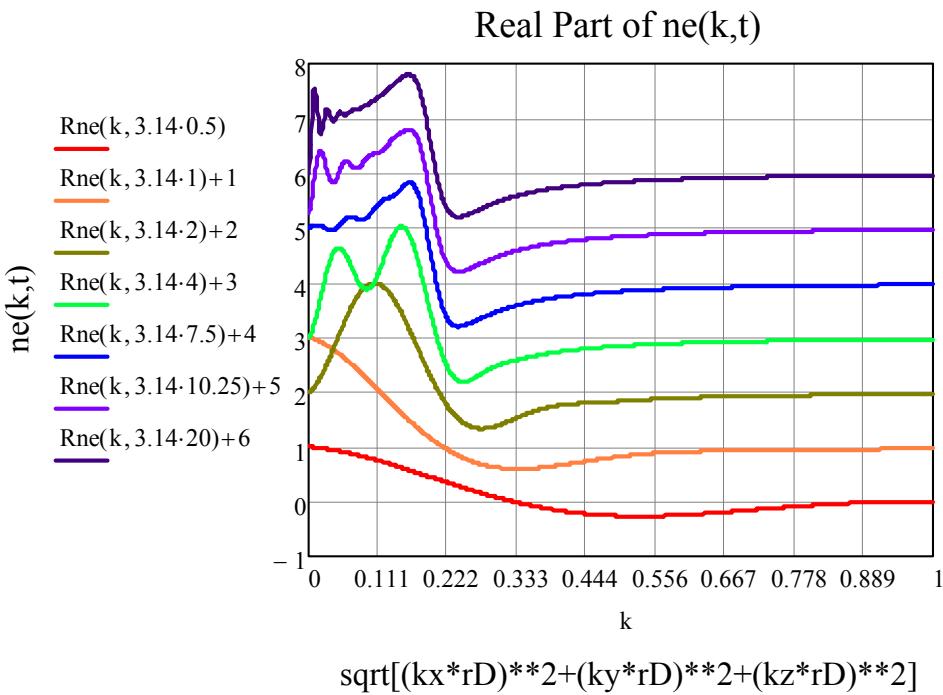
- The electron response to a certain wavelength is

$$\tilde{n}_l(\vec{k}, t) = \frac{\omega_p^2 Z_i}{\omega_p^2 + \lambda(\vec{k})^2} \left\{ 1 - \exp(\lambda(\vec{k}) \cdot t), \left[\cos(\omega_p t) - \frac{\lambda(\vec{k})}{\omega_p} \cdot \sin(\omega_p t) \right] \right\}$$

 Landau Damping

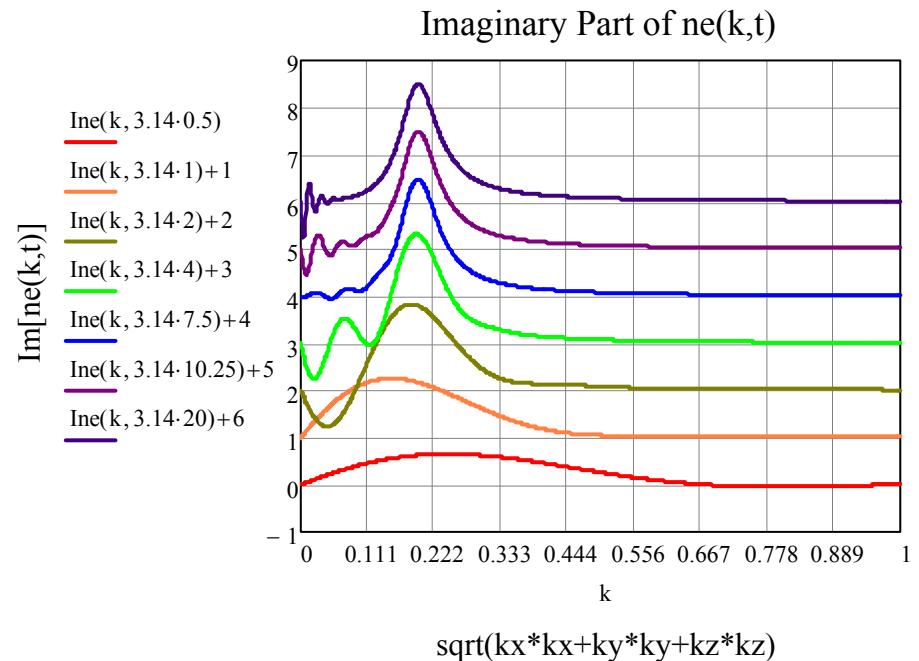
- The response drops for large wavelength.
- The Landau damping rate increases with wavelength and temperature.

Wavelength Depend. of the Electron Response



The wave vector k is along the moving direction of the ion.

$$\frac{\vec{k} \cdot \vec{v}}{kv} = 1$$



Solution in x space

The solution in x space can be obtained by Fourier transform.

$$\dot{\tilde{n}}_1(\vec{x}, t) = \frac{Z_i \omega_p t \sin(\omega_p t)}{\pi^2 \sigma_x \sigma_y \sigma_z \left(t^2 + \frac{(x + v_{0x} t)^2}{\sigma_x^2} + \frac{(y + v_{0y} t)^2}{\sigma_y^2} + \frac{(z + v_{0z} t)^2}{\sigma_z^2} \right)^2}$$

If one uses the normalized variable

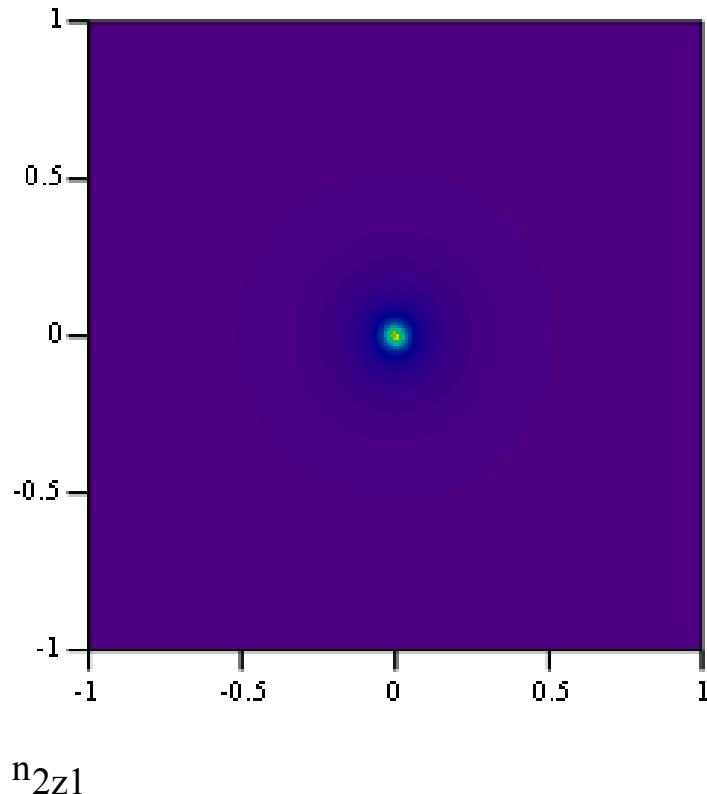
$$\psi \equiv \omega_p t \quad \bar{v}_{0i} = \frac{\vec{v}_{0i}}{\sigma_i} \quad \bar{x}_i = \frac{x_i}{r_{Di}}$$

the electron density fluctuation is

$$\tilde{n}_1(\vec{x}, t) = \frac{Z_i}{\pi^2 r_{Dx} r_{Dy} r_{Dz}} \int_0^\psi \frac{\psi_1 \sin(\psi_1) d\psi_1}{\left(\psi_1^2 + (\bar{x} + \bar{v}_{0x} \psi_1)^2 + (\bar{y} + \bar{v}_{0y} \psi_1)^2 + (\bar{z} + \bar{v}_{0z} \psi_1)^2 \right)^2}$$

which can be expressed into sum of sine integral.

Density Response for an Rest Ion



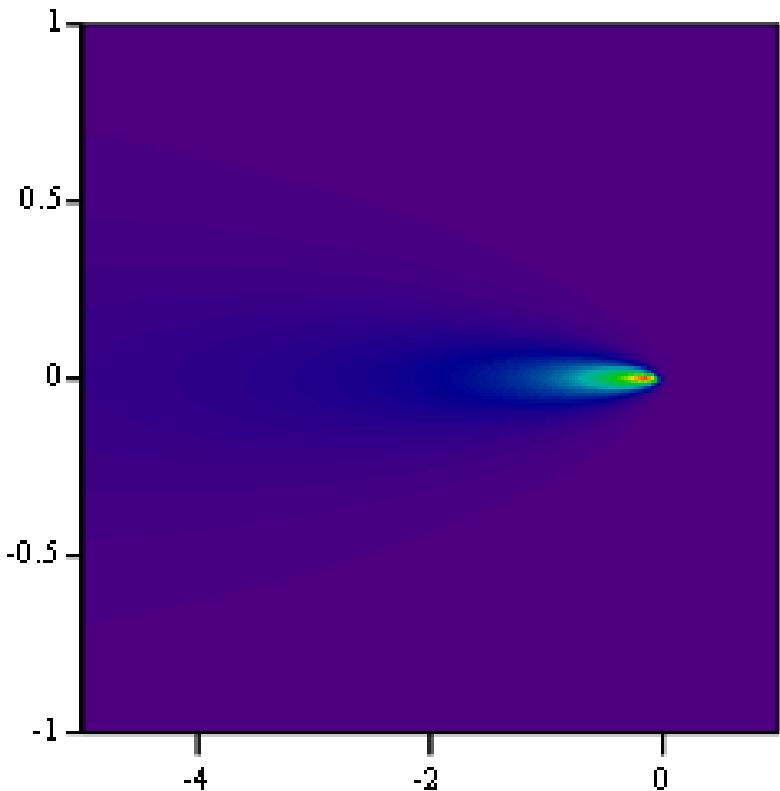
- The solution reduced to the well known Debye screening formula at the condition

$$t \rightarrow \infty \quad \vec{v}_0 = 0 \quad r_{Dx} = r_{Dy} = r_{Dz}$$

$$\begin{aligned}\tilde{n}_1(\vec{x}, t) &= \frac{Z_i}{\pi^2 r_D^3} \int_0^\infty \frac{\psi \sin(\psi) d\psi}{(\psi^2 + \bar{r}^2)^2} \\ &= \frac{Z_i}{4\pi r_D^2} \cdot \frac{1}{r} \exp\left(-\frac{r}{r_D}\right)\end{aligned}$$

$$\psi = \pi$$

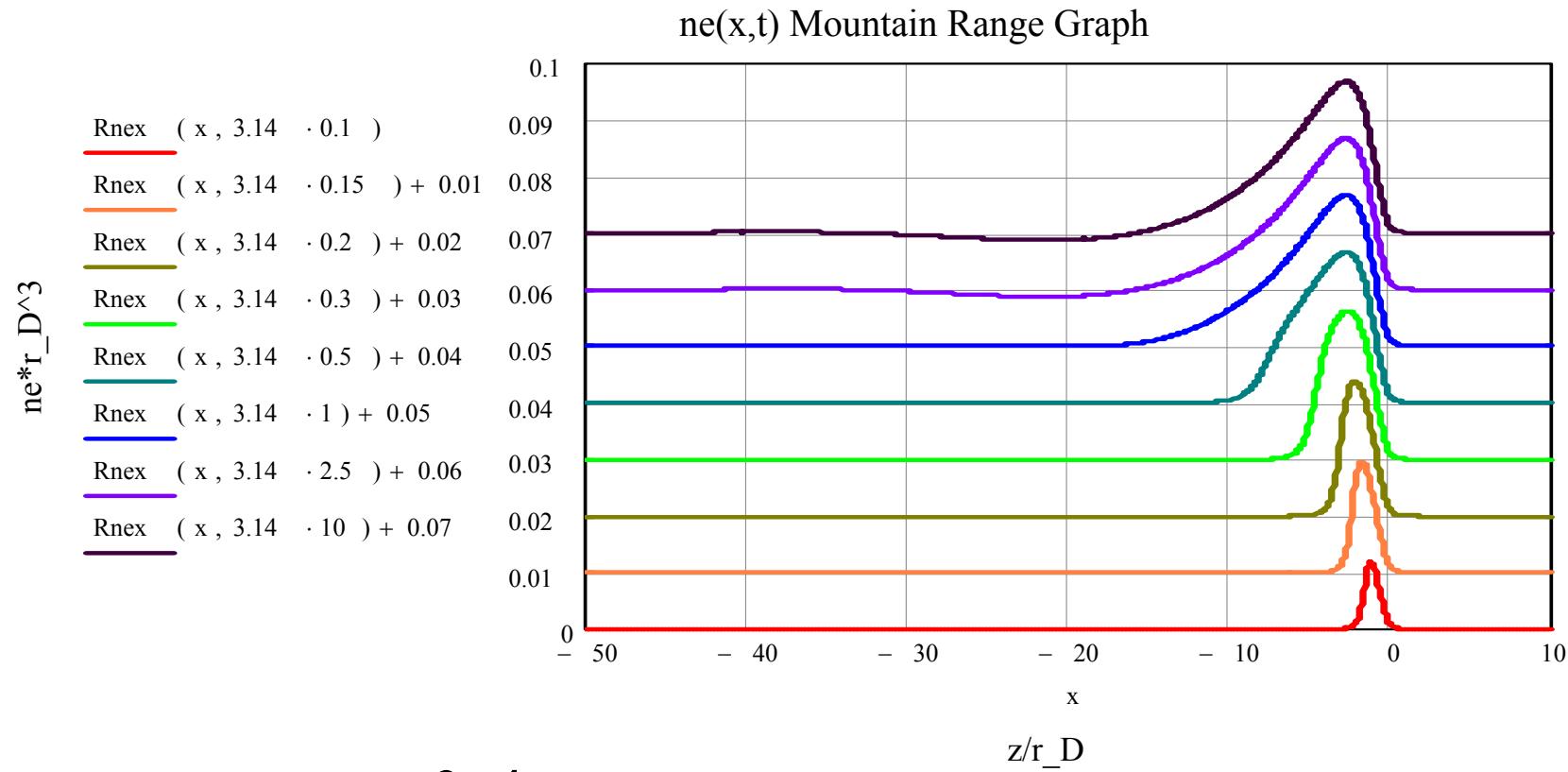
Density Response for an moving Ion



$$n_{2z1} \quad \psi = \pi \quad \bar{v}_0 = (0,0,10)$$

- The formula can describe the dynamics of the electron shielding for an ion with random 3D velocity in an infinite plasma (isotropic or anisotropic).

Time Dependence of Electron Response



$$x = y = 0.4r_D$$

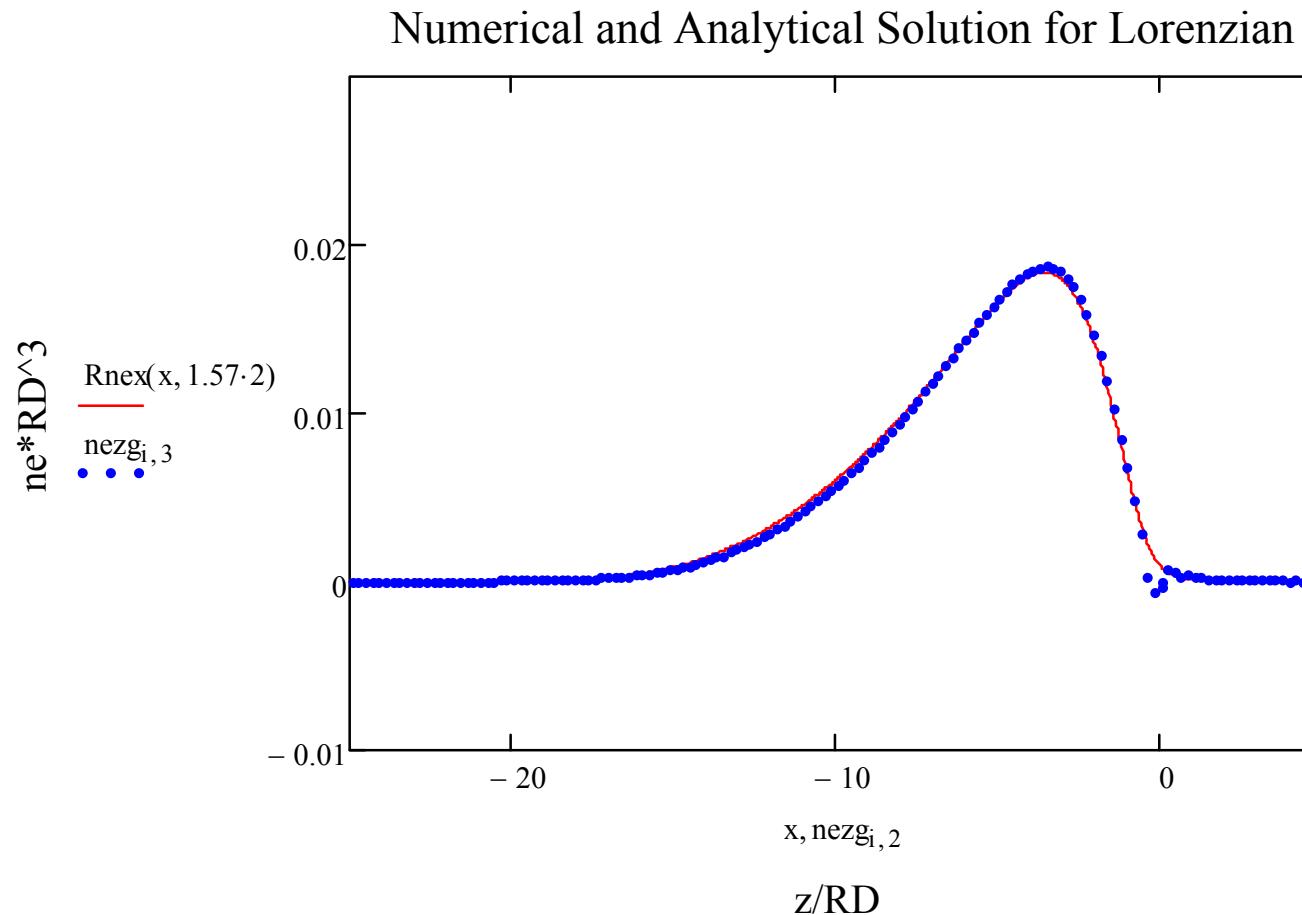
Numerical Solution (Gaussian Distribution)

- For general electron velocity distribution $f_0(\vec{v})$, the solution can be found by numerically solving the integral equation in k space and then Fourier transform into x space.

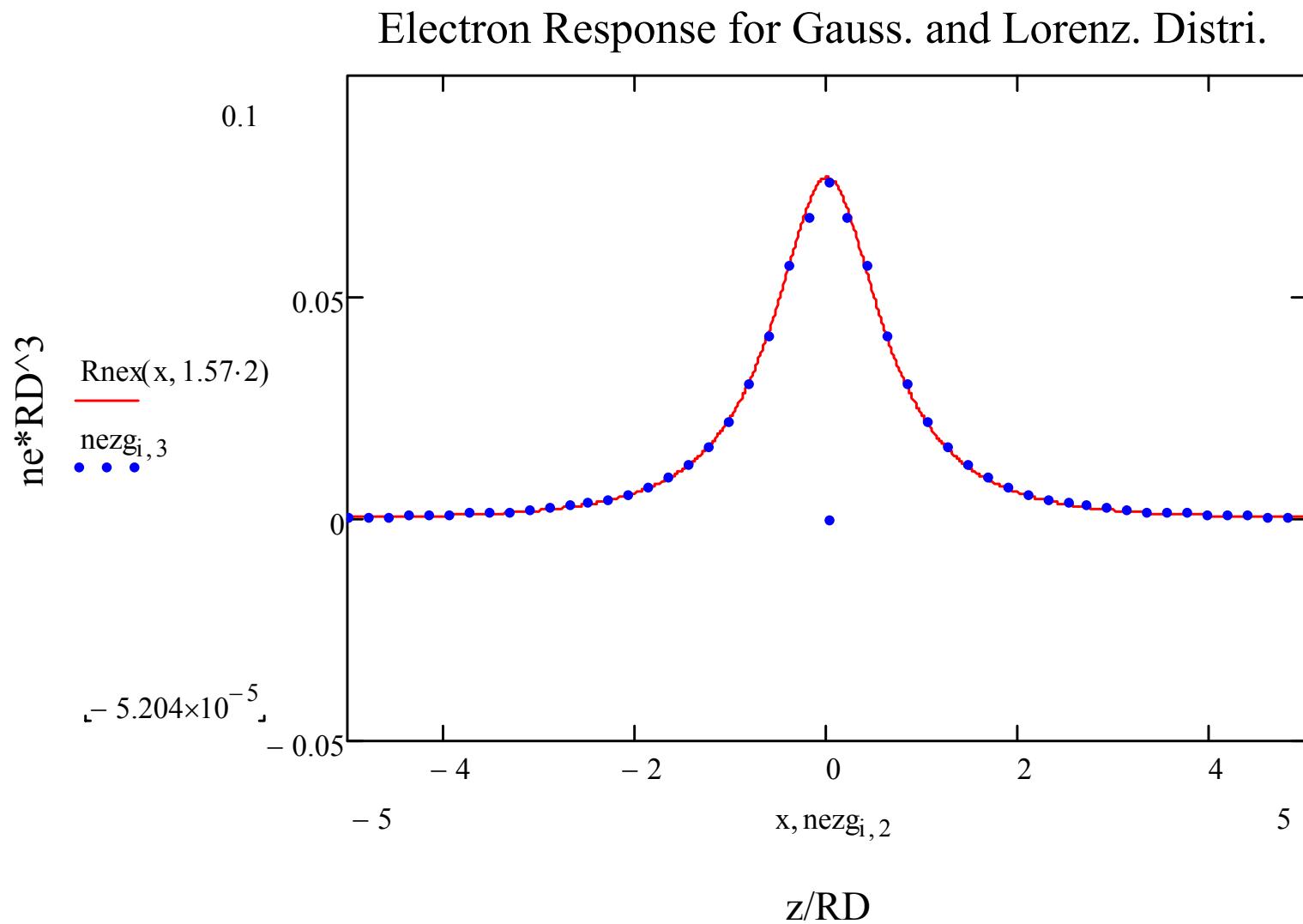
$$\tilde{n}_1(\vec{k}, t) = \omega_p^2 \int_0^t [\tilde{n}_1(\vec{k}, t_1) - Z_i] t_1 - t g(\vec{k}(t_1 - t)) dt_1$$

$$g(\vec{u}) \equiv \int f_0(\vec{v}) e^{i\vec{u} \cdot (\vec{v} + \vec{v}_0)} d^3 v$$

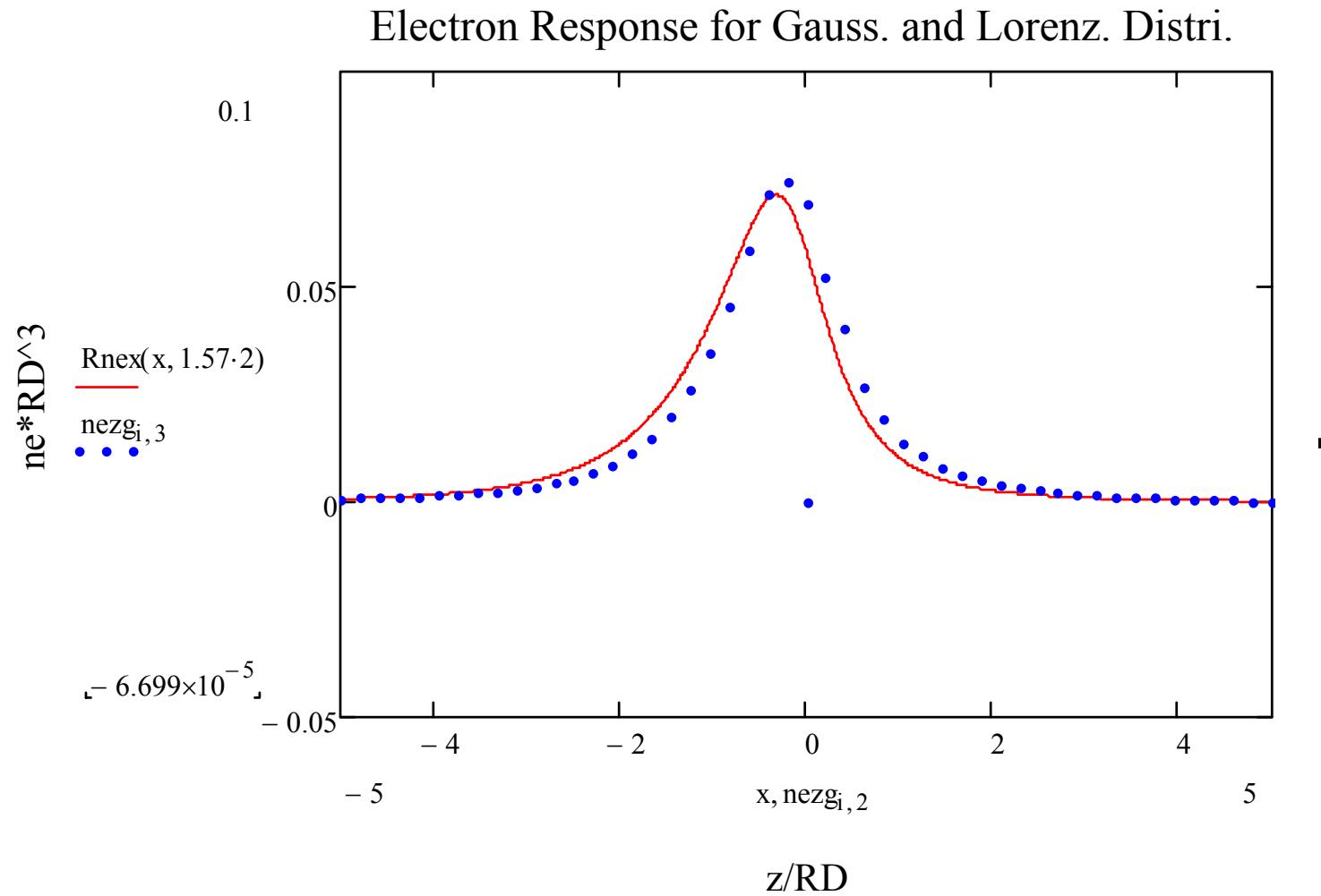
Debuging Codes With Formula $v_{i,z} = 5\sigma_s$ $\psi = \pi$



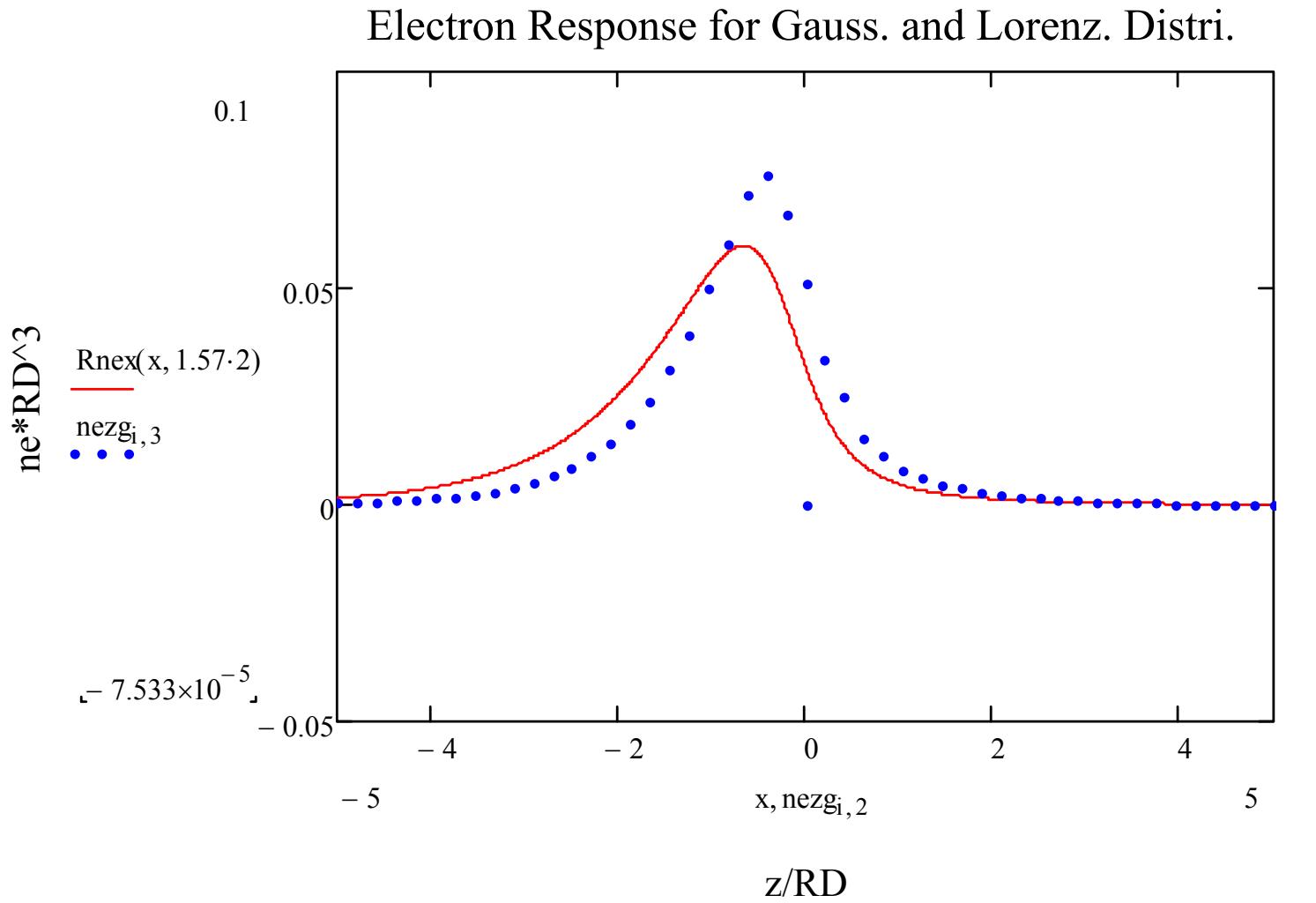
Comparison of Gauss.&Loren for $\nu_{i,z} = 0 \quad \psi = \pi$



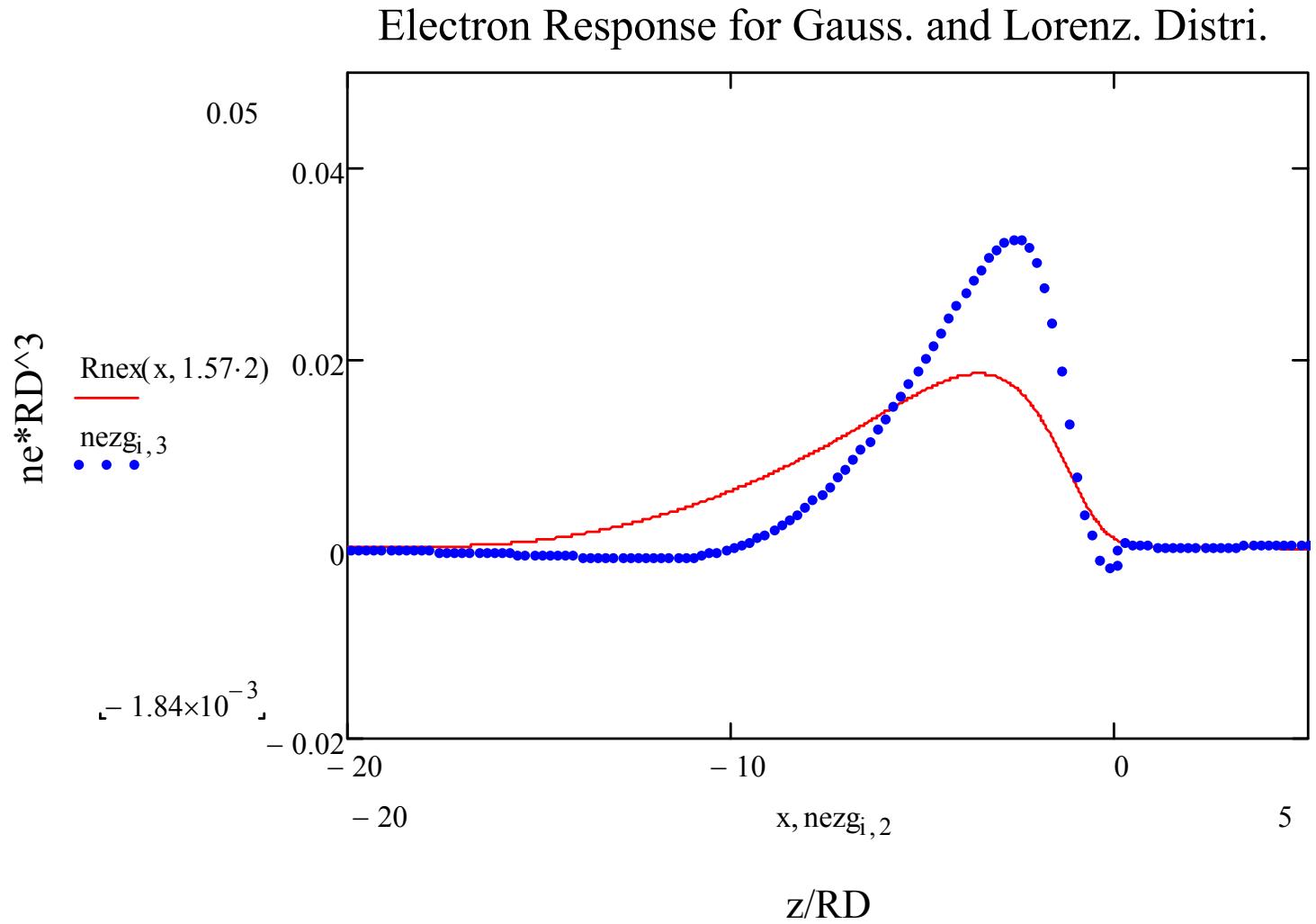
Comparison of Gauss.&Loren for $v_{i,z} = 0.5\sigma_s$ $\psi = \pi$



Comparison of Gauss.&Loren for $v_{i,z} = \sigma_s$ $\psi = \pi$

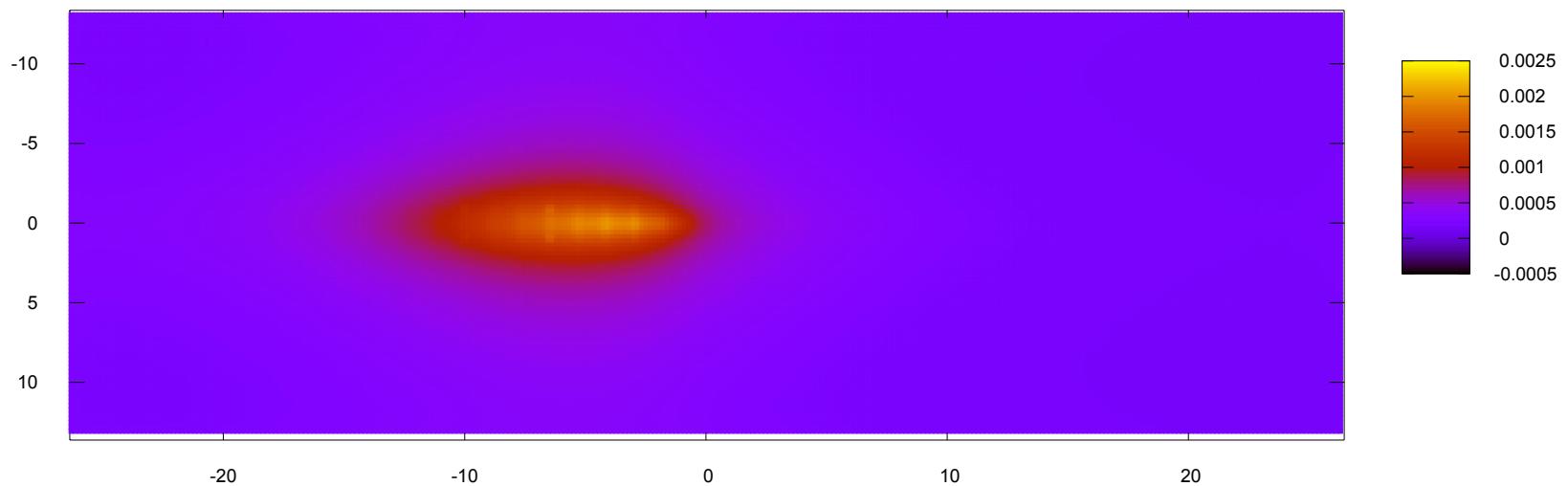


Comparison of Gauss.&Lorenz for $v_{i,z} = 5\sigma_s$ $\psi = \pi$



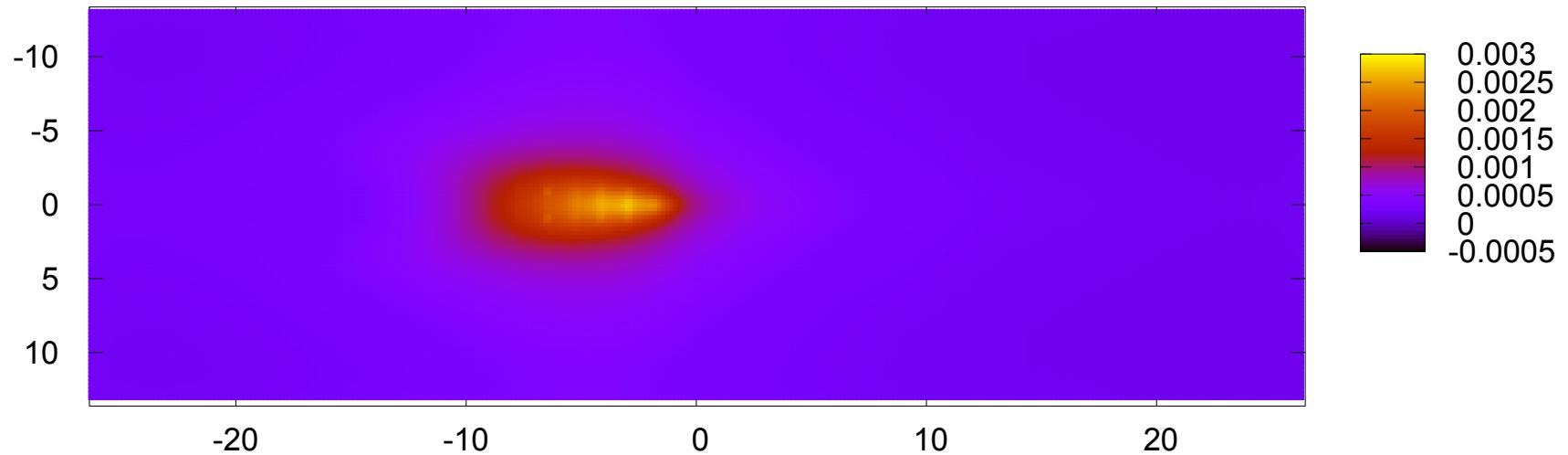
2D Mapping of Electron Density

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Lorenzian $v_z=5\sigma$, $x=y=0.41rD$

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Summary

- The analytical result reproduces the Debye screening formula for an rest ion and for slow ions($v_{0i} < \sigma_i$) it is close to the numerical Gaussian result. For fast ions, the Lorenzian approximation gives much smoother result than Gaussian distribution.
- The wavelength dependence of the electrons' response oscillates about one plasma oscillation period. After that, oscillation only remains for small wavelength due to Landau Damping.
- The numerical Gaussian solution might be relatively fast compared with the PIC simulation and can be used for a quick estimation.